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# DIFFRACTION OF ELECTROMAGNETIC WAVES BY A PLANE WIRE GRATING

by

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## ABSTRACT

Previous work on diffraction by a wire grating is briefly reviewed. The problem of diffraction by a finite grating consisting of perfectly conducting wires of arbitrary cross-section is formulated in terms of characteristic plane waves corresponding to the various order spectra defined in optics. Scattering matrix elements are expressed as stationary functionals of current distribution on the grating wires, for the incident wave falling at right angles to the grating elements and polarized either parallel or perpendicular to them. Similar expressions are obtained for the impedance matrix elements; these are evaluated for the thin wire grating, and an equivalent network constructed.

## 1. Introduction

The wire grating is commonly used to measure wavelength of infrared radiation. With the increased interest in the frequency range bridging the far infrared and microwaves, it is important to investigate the behavior of the grating when the wavelength is of the same order of magnitude as the grating constant.

In the infrared region the grating constant is much greater than a wavelength. In this case one can assume that there is no interaction between the wires of the grating. The diffraction field is taken as the sum of diffraction fields of the individual wires of the grating, the field of any one wire calculated neglecting the other wires[1]. The grating is assumed to consist of a set of infinite, parallel, equidistant, perfectly conducting circular cylinders. The set may be taken as finite, showing the background in addition to the various order spectra. Only the infinite grating will be considered here. This approach was extended somewhat by Twersky[2] who included the first order interaction effects between neighboring elements in the case of an acoustic wave incident on a grating of perfectly rigid circular cylinders.

Ignatowski took a completely different approach to the problem. In a number of papers[3] he developed a method for calculating the field from the current distribution. He then took up the problem of the infinite grating[4], and showed that due to the periodicity of the boundary conditions it is natural to perform a Fourier analysis of the field, thus splitting it into characteristic waves. These characteristic waves are precisely the various order spectra treated in standard books on optics. The current distribution on the wires is identical, on neighboring elements, except for a constant phase difference. The intensity and phase of each diffracted wave can then be calculated from the current distribution on a single wire and the phase difference. It remains to determine the current distribution. One can obtain good approximations to the current distribution very simply at wavelengths much smaller or much larger than wire thickness. These approximations, for the case of the circular cylinder can be found

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[1] See for example Schaefer & Reiche, Ein Beitrag zur Theorie der Gitterinversion, Ann. der Phys. 32, 577, 1910.

[2] V. Twersky, On the Theory of the Non-Specular Reflection of Sound, PhD.Dissertation, New York University, 1949.

[3] W.von Ignatowsky, Diffraction and Reflection Derived from Maxwell's Equations, Annalen der Physik 23, 875, 1907;25,99,1908;26,1031,1908.

[4] W.von Ignatowsky, On the Theory of the Grating, ibid, 44, 369, 1914.

in Ignatowsky's paper. Although better approximations to the current distribution can be obtained from an integral equation, calculations become cumbersome and lengthy.

There is another method of calculating diffracted wave amplitudes which was introduced by Schwinger and his co-workers at the Radiation Laboratory for the solution of waveguide problems. The variational method has been applied to a variety of scattering problems[5,6]. These, however, have been of two types: problems in which the propagation constant has a continuous spectrum[5], and those in which it has a discrete spectrum with a single imaginary value, the rest real[6]. This last restriction corresponds to the assumption of such dimensions of the waveguide that only the dominant mode is propagated without attenuation. If this assumption is dropped, the waveguide problem[7] is very closely related to the grating problem with no restriction of wavelength.

Just as in the case of a waveguide we can choose between the traveling wave representation of the field, and calculate the scattering matrix, or the standing wave representation - and calculate the impedance matrix. A thorough discussion of the impedance representation of the diffraction grating, with the restriction to a single propagated mode, can be found in a report by N.Marcuvitz[8]. Although the results of the calculations are listed there for a number of wire cross sections. no indication is given of the method used. According to Dr. Marcuvitz the integral equation was solved directly.[9]

The variational method has been used to calculate the surface impedance of a wire grating[10,11]. In this case the calculations give the total power transfer through the grating, rather than the intensities of the various diffracted waves. The relation between surface impedance and an impedance matrix element for the infinite grating is essentially the same as that between the total and differential scattering cross section of a body.

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[5] Schwinger & Levine, On the Theory of Diffraction by an Aperture in an Infinite Plane Screen. Phys. Res. 74, 958,4948; 75,1423,1949. See also W. Sollfrey, The Variational Solution of Scattering Problems, Research Report EM-11, Mathematics Research Group, N.Y.U., Contract No. AF-19(122)-42. Mr. Sollfrey suggested the method used here and in Ref. 7, and contributed in both cases by many helpful discussions.

[6] D.S. Saxon, Notes on Discontinuities in Waveguides, lectures by J.S.Schwinger.

[7] J. Greig, B. Fleishman, and J. Shmoys, unpublished work.

[8] N. Marcuvitz, Impedance Description of Planar Gratings, Rep.R-140-47,PIB-90.

[9] Contract No. WLENG, W28-099-2c-146, Polytechnic Research Institute,Brooklyn,N.Y., By personal communication.

[10] W.H.Watson, Physical Principles of Waveguide Transmission and Antenna Systems, Oxford University Press, 1947, pp.63-65.

[11] J.W.Miles, The Diffraction of Plane Wave Through a Grating, Quart. Appl.Math., 7, 45, 1949.

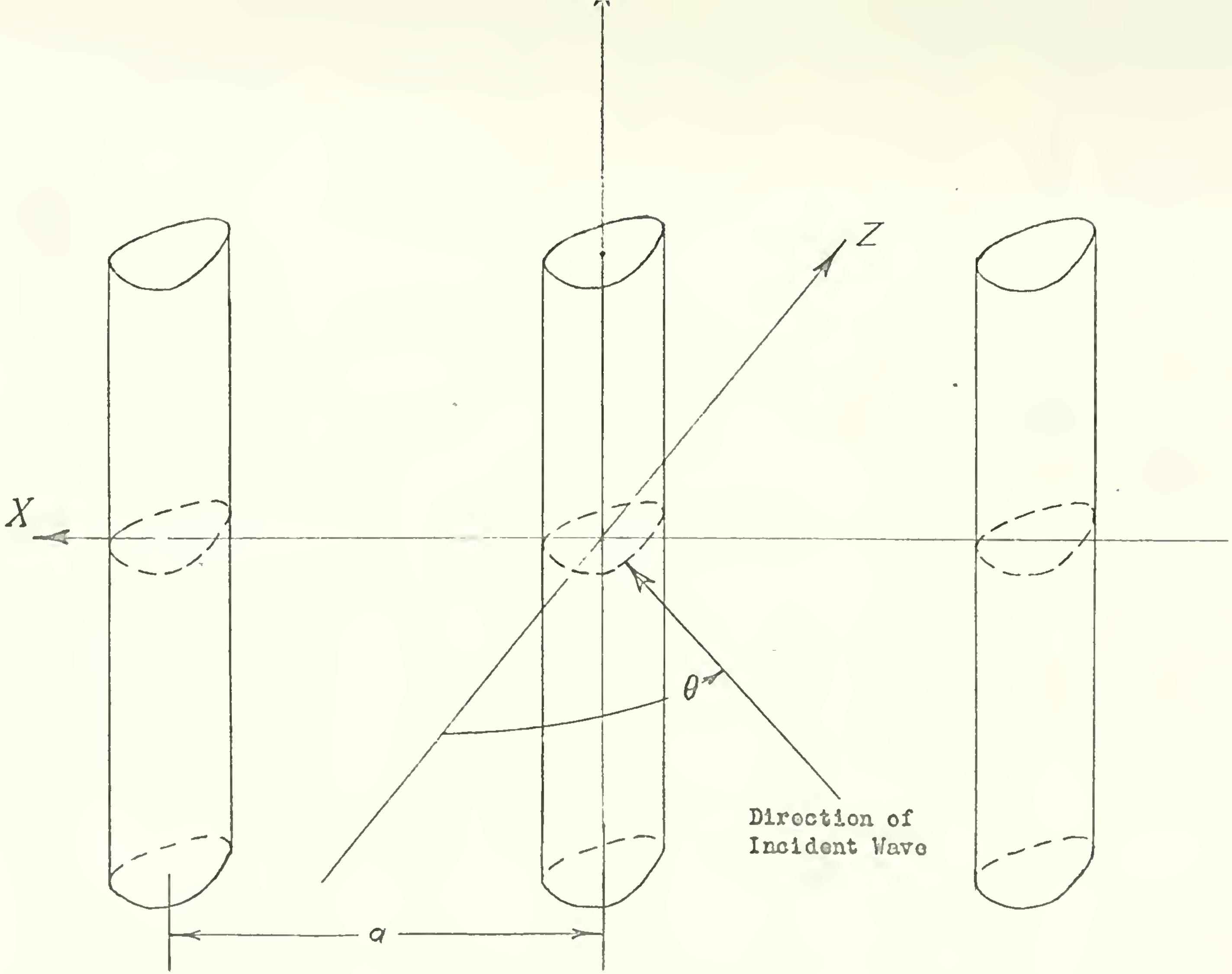

Fig. 1
The Wire Grating

In this report the infinite plane grating consisting of perfectly conducting wires of arbitrary cross section will be considered. The incident wave will be assumed to fall on the grating in a direction perpendicular to the wire, and to have arbitrary polarization. For this case the scattering matrix elements will be expressed as stationary functionals of current distribution on the grating wires. Similar expressions will be obtained for impedance matrix elements. The impedance matrix elements will be evaluated for the thin wire grating, and an equivalent network constructed. Calculations for more involved cases are in progress and will be presented in a future report.

## 2. Formulation of the Problem

It will be assumed that an electromagnetic wave is incident on the grating in a direction perpendicular to the grating elements (see Fig. 1). It was shown by Marcuvitz[8] that, for arbitrary polarization of the incident wave, the problem can be split into separate problems for each component of polarization.

1. Electric field vector of the incident wave is parallel to grating elements. In this case the electric field has only one component $E_y = \phi(x,z)$. This component satisfies the wave equation $\nabla^2\phi + (2\pi/\lambda)^2\phi = 0$, with $\phi$ vanishing on the perfectly conducting surface of the wires, and satisfying the Sommerfeld radiation condition, except in the direction of the incident wave.

2. Magnetic field vector of the incident wave is parallel to grating elements. In this case the magnetic field has only one component $H_y = \phi(x,z)$. This component satisfies the wave equation $\nabla^2\phi + (2\pi/\lambda)^2 \quad = 0$, with the normal derivative of $\phi$ vanishing on the surface of the wires, and satisfying the radiation condition except in the direction of the incident wave.

## 3. Field Representation in Traveling Waves

In case of either polarization the electromagnetic field can be represented as a sum of an infinite number of characteristic modes. Let the incident wave fall on the grating in a direction making an angle $\theta$ with the z-axis (see Fig. 1). This means that the x dependence of the incident wave is such that the phase difference between currents excited in neighboring grating elements is $2\pi a \sin\theta/\lambda$. The set of modes is then constructed by requiring the same phase difference between grating elements. It turns out that, in deriving stationary expressions, it is also necessary to consider the conjugate problem, in which

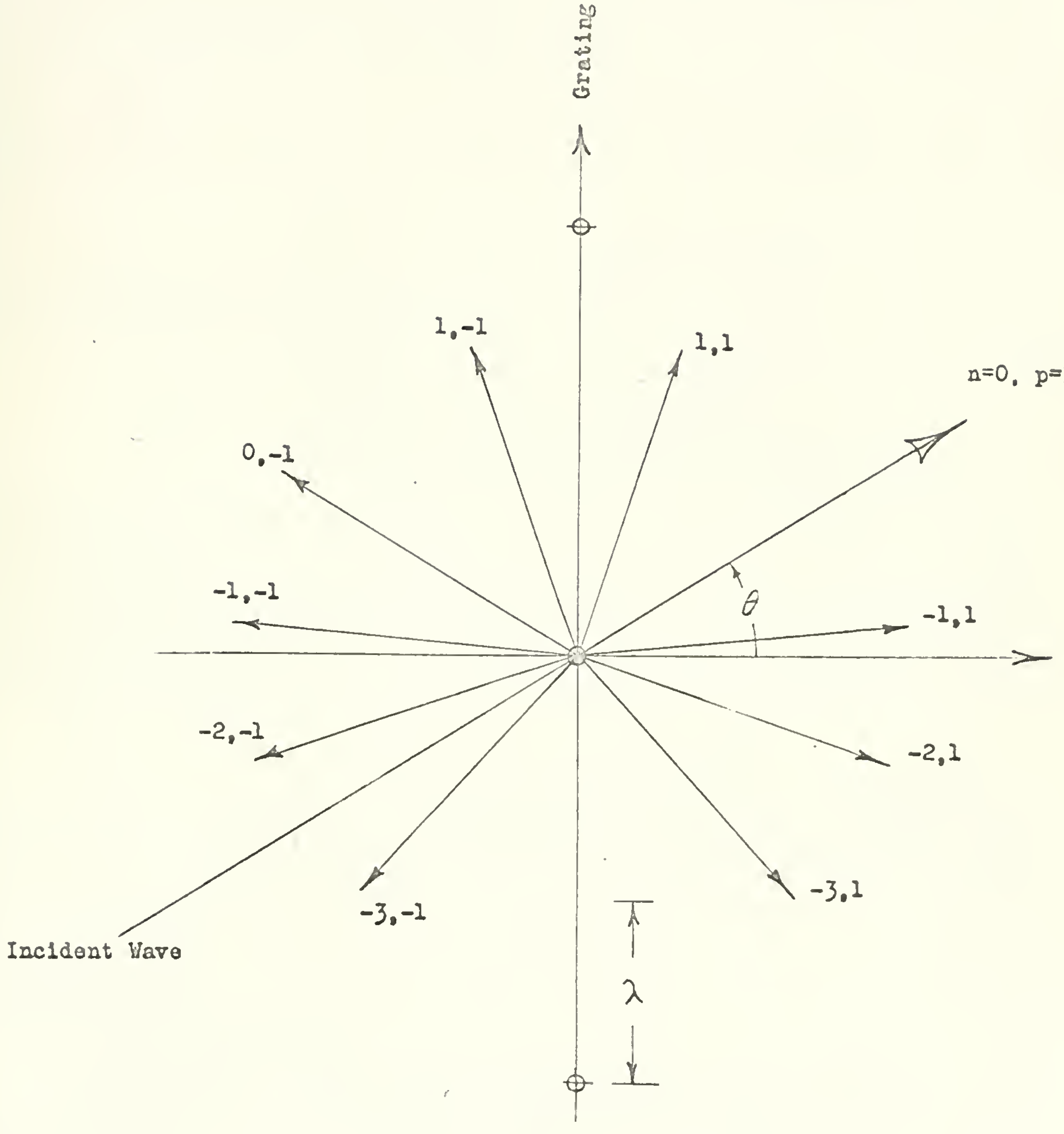

Fig. 2

Directions of Characteristic Waves

s = + 1

the phase difference is $-2\pi a \sin\theta/\lambda$. Thus, taking the time dependence to be $\exp(-i\omega t)$, the x dependence must be $\exp \pm i(2\pi \sin\theta/\lambda + 2\pi n/a)x$ and we have the complete traveling wave representation of $\phi$

$$\phi_{n,p,s} = \exp i\left[s(k\sin\theta + 2\pi n/a)x + p K_n z\right] \tag{1}$$

$$k = 2\pi/\lambda \qquad K_n^2 = k^2 - (k\sin\theta + 2\pi n/a)^2$$

where n-mode number, an arbitrary integer or zero

$p = \pm 1$, depending on whether the wave moves in the positive or negative z direction

$s = \pm 1$, the positive value corresponding to the original problem, and the negative to the conjugate one.

Only for a finite number of these modes $K_n$ is real: for other values of n the choice of the sign of one radical for $K_n$ is dictated by the requirement of finite fields far away from the grating. Thus for $|kz| \gg 1$, the attenuated modes can be neglected and the field consists of propagated modes only (see Fig. 2), i.e. those satisfying the inequality

$$-(1 + \sin\theta)\, a/\lambda < n < (1 - \sin\theta)\, a/\lambda \tag{2}$$

The special case in which either bound is integral will not be considered. In general, if any one mode $\phi_{n,p,s}$ is incident on the grating, then all other modes of the set $\phi_{n',p',s}$ will be excited.

## 4. T E Wave Problem.

### 4.1 The Green's Function

Let us define a Green's function for the first problem ($\phi = E_y$) the field at (x,z) due to a set of current element at $(x' + na,\ z')$, $-a/2 < x' < a/2$, of unit amplitude and differing in phase by $ska \sin\theta$. The function satisfies the inhomogeneous wave equation[12].

$$(\nabla^2 + k^2)\, G(s;\, x,z;\, x',z') = \sum_{n=-\infty}^{\infty} \exp(inska\sin\theta)\,\delta(x-x'-na)\ (z-z') \tag{3}$$

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[12] See ref. 6 p. 2, Section I.

The Green's function can be evaluated by expanding it into the set of modes defined in (1). Let

$$(4) \qquad G = \sum_{-\infty}^{\infty} \exp\left[si\,(k \sin\Theta + 2\pi n/a)\,(x-x')\right] f_n\,(z;\, x',z')\,.$$

Then

$$(4a) \quad G_{xx} = -\sum_{-\infty}^{\infty} (k \sin\Theta + 2\pi n/a)^2 \exp si\left[(k \sin\Theta + 2\pi n/a)(x-x')\right]\; f_n(z;x',z').$$

and

$$(4b) \quad (\nabla^2 + k^2)G = \sum_{-\infty}^{\infty} (\partial^2/\partial z^2 + K_n^2) \exp\left[si(k \sin\Theta + 2\pi n/a)(x-x')\right]\; f_n(z;x',z')\,.$$

Substituting the expression from (3) for the left-hand member of (4b), then multiplying both sides by $\exp\left[-si\,(k \sin\Theta + 2\lambda n/a)x\right]$ and integrating with respect to x from -a/2 to a/2

$$(5) \qquad (\partial^2/\partial z^2 + K_n^2)\, f_n\,(z;x',z') = \delta\,(z-z')/a$$

Then, imposing the Sommerfeld radiation condition, and requiring continuity at $z = z'$

$$(6) \qquad f_n = A_n \exp\,(i\,K_n\,|z-z'|)$$

Integrating (5) with respect to z from z'- to z'+, and making use of (6)

$$(7) \qquad A_n = i/2aK_n$$

and finally

$$(8)\; G(s;x,z;x',z') = (i/2a) \sum_{-\infty}^{\infty} K_n^{-1} \exp\left[si\,(k \sin\Theta + 2\pi n/a)(x-x') + iK_n|z-z'|\right]$$

### 4.2 Integral Equation

The total field is equal to the sum of the incident field and that generated by the surface currents on the wires of the diffraction grating. Since the grating is infinite, the current distribution on each wire is identical, thus permitting us to sum the contributions of the corresponding current filaments on

all wires, and then to integrate around a single wire, say one located between $x = -a/2$ and $x = a/2$. The summation has been already performed in the derivation of the Green's function. Let the incident field consist of a single mode n;p,s. Denote by $I_{n,p,s}(x,z)$ the current distribution excited by this field in the central cylinder of the grating. Then we have

$$(9) \qquad \phi = \phi_{inc} + \phi_{sc} = \phi_{n,p,s}(r) + \int G(s;r;r')\, I_{n,p,s}(r')\, dr'$$

where $r \equiv x,z$ and the integral is taken around the wire. The current distribution is obtained by requiring $\phi$ to vanish on the obstacle. The integral equation for I is then,

$$(10) \quad \text{for } r \text{ on wire} \qquad \phi_{n,p,s}(r) = -\int G(s;r;r')\, I_{n,p,s}(r')\, dr'$$

### 4.3 The Scattering Matrix

Let the incident field be

$$(11) \qquad \phi_{inc} = \sum_{n=-i}^{j} \sum_{p} A_{n,p,e}\, \phi_{n,p,s}$$

the summation being over all propagated modes ($n=-i,\ldots,j$; $p= -1,+1$). The scattered field at large distance from the grating also consists of the propagated modes, with the same value of s.

$$(12) \qquad \phi_{sc} = \sum_{m=-i}^{j} \sum_{p} B_{m,p,s}\, \phi_{m,p,s}$$

The scattering matrix is defined as the operator which transforms the given vector $\bar{A}$ into $\bar{B}$

$$\bar{B} = \bar{\bar{S}}\ \bar{A}$$

or, in components

$$(13) \qquad B_{m,g,s} = \sum S_{mgs \leftarrow nps}\, A_{nps}$$

In order to evaluate the elements of $\bar{\bar{S}}$, take, as in Eq. (9), a vector $\bar{A}$ consisting of a single component and of unit amplitude, and evaluate the components of $\bar{B}$. For that purpose let us investigate the asymptotic value of the Green's function far

away from the grating. It is clear that for $k|z' - z| \gg 1$ all attenuated mode contributions in (8) can be neglected and

(14a) $$z \to \infty \quad G(s;x,z;x',z') \to (i/2a) \sum_{m=-i}^{j} K_m^{-1} \exp\left[si(k \sin \Theta + 2\pi m/a)(x-x') + iK_m(z-z')\right] =$$

$$= (i/2a) \sum_{m=-i}^{j} K_m^{-1} \phi_{m,+1,s}(x,z)\, \phi_{m,-1,-s}(x',z')$$

$$z \to -\infty \quad G(s;x,z;x',z') \to (i/2a) \sum_{m=-i}^{j} K_m^{-1} \exp\left[si(k \sin \Theta + 2\pi m/a)(x-x') + iK_m(z'-z)\right] =$$

(14b) $$= (i/2a) \sum_{m=-i}^{j} K_m^{-1} \phi_{m,-1,s}(x,z)\, \phi_{m,+1-s}(x',z')$$

The asymptotic form of the scattered field is then

$$\phi_{sc} = + \int G(s;r;r')\, I_{n,p,s}(r')\, dr'$$

(15a) $$z \to \infty \quad \phi \to + (i/2a) \sum_{m=-i}^{j} K_m^{-1} \phi_{m,+1,s}(r) \int \phi_{m,-1,-s}(r')\, I_{n,p,s}(r')\, dr'$$

(15b) $$z \to -\infty \quad \phi \to + (i/2a) \sum_{m=-i}^{j} K_m^{-1} \phi_{m,-1,s}(r) \int \phi_{m,+1,-s}(r')\, I_{n,p,s}(r')\, dr'$$

(16) $$S_{m,q,s \leftarrow n,p,s} = + (i/2aK_m) \int \phi_{m,-q,-s}(r')\, I_{n,p,s}(r')\, dr'$$

Similarly

(17) $$S_{n,-p,-s \leftarrow m,-q,-s} = +(i/2aK_n) \int \phi_{n,p,s}(r')\, I_{m,-q,-s}(r')\, dr'$$

Multiplying both sides of Eq. (10) by $I_{m,-g,-s}(r)$ and integrating around the wire

(18) $$-\int \phi_{n,p,s}(r)\, I_{m,-q,-s}(r)\, dr = \iint I_{m,-q,-s}(r)\, G(s;r;r')\, I_{n,p,s}(r')\, dr'\, dr$$

Combing (16) and (18)

$$(19) \qquad K_m\, S_{m,q,s \leftarrow n,p,s} = \frac{-i}{2a}\, \frac{\int \phi_{m,-q,-s}(r')\, I_{n,p,s}(r')\, dr' \int \phi_{n,p,s}(r)\, I_{m,-q-s}(r)\, dr}{\iint I_{m,-q,-s}(r)\, G(s;r;r')\, I_{n,p,s}(r')\, dr'dr}$$

and from (17) we get

$$(20) \qquad K_n\, S_{n,-p,-s \leftarrow m,-q,-s} = \frac{-i}{2a}\, \frac{\int \phi_{m,-q,-s}(r)\, I_{n,p,s}(r) dr \int \phi_{n,p,s}(r')\, I_{m,-q,-s}(r') dr'}{\iint I_{n,p,s}(r)\, G(-s;r;r') I_{m,-q-s}(r')\, dr'dr}$$

From (19) and (20) we can derive a reciprocity relation. It can be easily verified that, due to the properties of the Green's function, the right-hand members of (19) and (20) are equal. Hence

$$(21) \qquad K_m\, S_{m,q,s \leftarrow n,p,s} = K_n\, S_{n,-p,-s \leftarrow m,q,s}$$

The main advantage of expression (19) is that it is stationary with respect to variation of the current distributions $I_{n,p,s}$ and $I_{m,-q-s}$ about the distributions satisfying the integral equation (10). It can be readily shown[13] that substituting $I^o_{n,p,s} + \epsilon I'_n$ for $I_{n,p,s}$, and $I^o_{m,-q-s} + \delta I'_m$ for $I_{m,-q-s}$, then expanding s as a power serves in $\epsilon$ and $\delta$, the coefficients of the linear terms, $\epsilon$ and $\delta$, vanish if $I^o_{n,p,s}$ and $I^o_{m,-q-s}$ satisfy the integral equation (10). By substituting approximate current distributions in (19), one can obtain very good estimates of the elements of the scattering matrix.

## 5. T M Wave Problem

The second problem, i.e. one in which the magnetic field of the incident wave is parallel to the grating elements, differs from the previous one only in the Green's function and the boundary condition to be satisfied. In the T E problem the magnetic field was in the xz-plane so that the currents excited in the grating elements were parallel to the y axis. The Green's function was the sum of the radiation fields of a set of current elements. The radiation field of a single current element, a line current parallel to the y axis is clearly cylindrically

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[13] See W. Sollfrey loc. cit. ref. 5, page 5

symmetric. In the present case we deal not with the radiation field of a line current, but with that of a uniform distribution, on a line parallel to the y axis, of short current elements perpendicular to the y axis, all in the same direction and in phase. Let $g(r,r')$ be the radiation field of a single current element in the T E problem; then $\frac{\partial}{\partial \nu'} g(r,r')$, the derivative of g with respect to the primed coordinates, in a direction perpendicular both to the y axis and the direction of the current flow, is the radiation field in the T M problem. Although the above discussion applied to the radiation field of a single current element, it is immediately applicable to the field generated by currents on a set of lines, distance a apart, with the appropriate phase difference between successive lines. Thus we have

$$(22) \qquad \Gamma(s;x,z;x',z') = \frac{\partial}{\partial \nu'} G(s;x,z;x',z')$$

where G is defined by Eq. (8). The field is then

$$(23) \qquad \phi = \phi_{inc} + \phi_{sc} = \phi_{n,p,s}(r) + \int G_{\nu'}(s;r;r')\, I_{n,p,s}(r')\, dr'$$

where G is differentiated in a direction normal to the wire surface. Since the normal derivative of tangential magnetic field vanishes on the wires, we have the integral equation

$$(24) \quad r \text{ on wire} \qquad \frac{\partial}{\partial \nu} \phi_{n,p,s}(r) = -\int G_{\nu\nu'}(s;r;r')\, I_{n,p,s}(r')\, dr'$$

By considering the asymptotic form of the Green's function given in (14), we obtain, as before, the elements of the scattering matrix

$$(25) \qquad S_{m,q,s \leftarrow n,p,s} = +(i/2a\, K_m) \int \left[\frac{\partial}{\partial \nu'} \phi_{m,-q,-s}(r')\right] I_{n,p,s}(r')\, dr'$$

From (24) we have

$$(26) \quad \int \left[\frac{\partial}{\partial \nu} \phi_{n,p,s}(r)\right] I_{m,-q-s}(r)\, dr = -\iint I_{m,-q-s}(r)\, G_{\nu\nu'}(s;r;r')\, I_{n,p,s}(r')\, dr'dr$$

and, finally

$$(27)\quad K_m S_{m,q,s \leftarrow n,p,s} = (-i/2a)\,\frac{\int\left[\frac{\partial}{\partial\nu'}\phi_{m,-q,-s}(r')\right]I_{n,p,s}(r')\,dr'\int\left[\frac{\partial}{\partial\nu}\phi_{n,p,s}(r)\right]I_{m,-q,-s}(r)\,dr}{\iint I_{m,-q,-s}(r)\;G_{\nu\nu'}(s;r;r')\;I_{n,p,s}(r')\,dr'dr}$$

The above expression has the same stationary properties as (20).

## 6. Standing Wave Representation and Impedance Matrix

Another form of representation of the scattering properties of a grating is by means of the impedance matrix rather than the scattering matrix. This is the standing wave representation, as opposed to the traveling wave representation used in the calculation of the scattering matrix. For simplicity it will be assumed that the grating is symmetric about the xy plane. This implies that

$$(28)\qquad S_{m,q,s \leftarrow n,p,s} = S_{m,-q,s \longleftarrow n,-p,s}$$

The modes will now be defined by the fields

$$(29a)\qquad \phi_{n,s} = V_{n,s}(z)\ \exp is\,(k \sin\Theta + 2\pi n/a)\,x$$

$$(29b)\qquad V_{n,s}(z) = A_{n,s} \sin K_n z + B_{n,s} \cos K_n z$$

Far away from the grating the field will approach asymptotically the sum of propagated modes, so that the far field is completely described by the set of coefficients $A_{ns}$, $B_{ns}$, $n = -i,\ldots,0,\ldots,j$. The far fields on the two sides of the grating are, in general, different, so that we shall use superscripts 1 and 2 to denote the coefficients $z \to -\infty$ and $z \to +\infty$ respectively.

At this point we shall make one more step in the direction of equivalent transmission line formalism. This formalism is natural for problems involving transmission of electromagnetic energy in a finite set of modes. The concepts of voltage and current follow directly from transverse electric and magnetic fields. The transmission line equations are then a consequence of Maxwell's equations directly. This formalism applied to the problem of the diffraction grating was discussed in detail by Marcuvitz[7], and only the bare essentials will be restated here.

As before we distinguish between the two problems arising from different polarizations of the incident wave. In the T E we deal with an electromagnetic field having components $E_y$, $H_x$, and $H_z$ only. The relevant Maxwell's equations, with time dependence $\exp(-i\omega t)$ assumed, are

$$(30a) \qquad (ik/\eta)\, H_x = -\frac{\partial E_y}{\partial z}$$

$$(30b) \qquad (ik/\eta)\, H_z = \frac{\partial E_y}{\partial x}$$

$$(30c) \qquad -ik\eta E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

Now $E_y$ and $H_x$, the transverse fields, must have the x dependence given in (29); in the $n^{th}$ mode

$$(31a) \qquad E_y = V_{n,s}(z) \;\exp\; is\,(k \sin\theta + 2\pi n/a)\, x$$

$$(31b) \qquad H_x = -I_{n,s}(z) \exp\; is\,(k \sin\theta + 2\pi n/a)\, x$$

and, from (30b)

$$(31c) \qquad H_z = \eta\; s(\sin\theta + 2\pi n/ka) \exp\; is\,(k \sin\theta + 2\pi n/a)\, x$$

Substituting (31) in (30), we obtain the transmission line equations

$$(32a) \qquad \frac{\partial}{\partial z} V_{n,s}(z) = (ik/\eta)\, I_{n,s}(z) \equiv (iK_n/Y_n)\, I_{n,s}(z)$$

$$(32b) \qquad \frac{\partial}{\partial z} I_{n,s}(z) = (i\eta/k)\left[k^2 - (k \sin\theta + 2\pi n/a)^2\right] V_{n,s}(z) \equiv iK_n Y_n V_{n,s}(z)$$

$$(32c) \qquad K_n = \left[k^2 - (k \sin\theta + 2\pi n/a)^2\right]^{1/2}$$

$$(32d) \qquad Y_n = \eta\, K_n/k$$

The field in a given mode is then completely specified when $V_{n,s}\,(z)$ and $I_{n,s}\,(z)$ are given for some value of $z = z'$. Although $z'$ is arbitrary, its choice being determined by convenience, let us take here for definiteness $z' = 0$ for the representation of fields on both sides of the grating. Let $V^{\,p}_{n,s} = V^{\,p}_{n,s}(0) = I^{\,p}_{n,s}(0)$; and define vectors $\bar{V} = \left\{ V^{\,p}_{n,s} \right\}$ and $\bar{I} = \left\{ I^{\,p}_{n,s} \right\}$ ; the problem now reduces to finding either the matrix Z or its inverse, Y.

(33a) $$\bar{V} = \bar{\bar{Z}}\ \bar{I}$$

(33b) $$\bar{I} = \bar{\bar{Y}}\ \bar{V}$$

In special cases Z or Y may be singular thus making the inverse representation impossible.

Since the grating is assumed to be symmetric about the $z = 0$ plane, the problem can be simplified. The matrix Y (or Z), whose rank is twice the number of propagated modes, r, can be decomposed into four matrices of rank $\nu$. Instead of using single vectors V and I, introduce voltage and current vectors on two sides of the grating

$$\bar{V}^p_s = \left\{ V^{\,p}_{n,s} \right\} \quad \text{and} \quad \bar{I}^p_s = \left\{ I^{\,p}_{n,s} \right\}, \text{ then}$$

(33a) $$\bar{I}^1_s = \bar{\bar{Y}}^{11}\,\bar{V}^{\prime}_s + \bar{\bar{Y}}^{12}\,\bar{V}^2_s$$

(33b) $$I^2_s = \bar{\bar{Y}}^{21}\,\bar{V}^{\prime}_s + \bar{\bar{Y}}^{22}\,\bar{V}^2_s$$

Due to the symmetry of the grating $\bar{\bar{Y}}^{11} = \bar{\bar{Y}}^{22}$. Also, from conservation of energy $\bar{\bar{Y}}^{12} = \bar{\bar{Y}}^{21}$. We shall now consider two cases

1. The field is even, $V^{\prime}_s = V^2_s$, $I^{\prime}_s = I^2_s$

(34) $$I^e_s = (Y^{11} + Y^{12})\ V^e_s$$

The problem is reduced to the half space $z \leq 0$ and we have the additional boundary condition $\dfrac{\partial \phi}{\partial z} = 0$, $z=0$.

2. The field is odd. $V_s^1 = -V_s^2$ , $I_s^1 = -I_s^2$

(35) $$I_s^o = (Y^{11} - Y^{12})\ V_s^o$$

In this case the additional boundary condition is $\phi = 0$, $z = 0$

Similarly, we have, in terms of the impedance matrix

(33c) $$V_s^1 = Z^{11} I_s^1 + Z^{12} I_s^2$$

(33d) $$V_s^2 = Z^{12} I_s^1 + Z^{22} I_s^2$$

(34a) $$V_s^e = (Z^{11} + Z^{12})\ I_s^e$$

(35a) $$V_s^o = (Z^{11} - Z^{12})\ I_s^e$$

We also have the characteristic modes in the standing wave representation, with the additional boundary condition at $z = 0$.

(36) $$\phi^e_{n,s} = \frac{1}{2}(\phi_{n,+1,s} + \phi_{n,-1,s})$$

(37) $$\phi^o_{n,s} = \frac{1}{2}(\phi_{n,+1,s} - \phi_{n,-1,s})$$

In what follows the term "incident field" will be understood to denote the imperturbed standing wave pattern given by a linear combination of $\phi^e_{n,s}$ or of $\phi^o_{n,s}$ .

## 7. Integral Equation and Stationary Form of Impedance Matrix Elements

We now proceed to derive the Green's functions for the even and odd fields, and their asymptotic forms. Since the Green's function for free space was derived previously, the additional boundary condition can be satisfied by the image method. For the even case we have

(38) $$\begin{aligned} G^e(s;r;r') &= G(s;x,z;x',z') + G(s;x,z;x',-z') = \\ &= (i/2a) \sum_{m=-\infty}^{\infty} K_m^{-1} \exp\left[ si\ (k \sin \Theta + 2\pi m/a)(x-x') \right] \left( \exp iK_m |z-z'| + \exp iK_m |z+z'| \right) \end{aligned}$$

so, in the region under consideration, $z \leq 0$

$$(39a)\quad G^{e}(s;r;r') = (i/a)\sum_{-\infty}^{\infty} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\exp(-iK_m z)\cos K_m z' \quad z \leq z'$$

$$(39b)\quad = (i/a)\sum_{-\infty}^{\infty} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\exp(-iK_m z')\cos K_m z \quad z \geq z'$$

The asymptotic form is obtained from (39a)

$$(40)\quad z \to -\infty\ G^{e}(s;r;r') \to (i/2a)\sum_{-i}^{j} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\left\{\exp(iK_m z') + \exp(-iK_m z')\right\}\exp(-iK_m z)$$

$$= (i/2a)\sum_{-i}^{j} K_m^{-1}\phi_{m,-1,s}(r)\left[\phi_{m,+1,-s}(r') + \phi_{m,-1,-s}(r')\right]$$

and for the odd case

$$(41)\quad G^{o}(s;r;r') = G(s;x,z;x,z) - G(s;x;z;x,-z) =$$

$$= (i/2a)\sum_{-\infty}^{\infty} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\left(\exp iK_m|z-z'| - \exp iK_m|z+z'|\right)$$

$$(42a)\quad G^{o} = (-1/a)\sum_{-i}^{j} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\exp(-iK_m z)\sin K_m z' \quad z \leq z'$$

$$(42b)\quad G^{o} = (-1/a)\sum_{-i}^{j} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\exp(-iK_m z')\sin K_m z \quad z \geq z'$$

and the asymptotic form is

$$G^{o} \to (i/2a)\sum_{-i}^{j} K_m^{-1}\exp\left[si(k\sin\theta + 2\pi m/a)(x-x')\right]\left\{\exp iK_m(z'-z) - \exp iK_m(-z'-z)\right\} =$$

$$(43)\quad = (i/2a)\sum_{-i}^{j} K_m^{-1}\phi_{m,-1,s}(r)\left[\phi_{m,+1,s}(r') - \phi_{m,-1,-s}(r')\right]$$

## Even Case

In the even case we have

$$\phi = \phi_o(r) + \int G(s;r;r')\, I_o(r')\ dr'$$

$$(44)\quad z \to -\infty\ \phi = \phi_o(r) + (i/2a)\sum_{-i}^{j} K_m^{-1}\phi_{n,-1,s}(r)\int\left[\phi_{n,+1,-s}(r') + \phi_{n,-1-s}(r')\right] I_o(r')dr')$$

Let (45) $$\phi_o(r) = \sum_{-i}^{j} a_n \phi_{n,s}^{e}$$

Furthermore, due to linearity of the problem, the current $I_o$ can be resolved into components depending on propagated mode intensities.

Let then

(46) $$I_o(r) = \sum_{-i}^{j} V_{n,s}^{e} \, \mathcal{I}_{n,s}^{e}(r)$$

From (44) and (45), the n^th^ mode voltage is

(47) $$V_{n,s}^{e}(z) = a_n \cos K_n z + (i/aK_n) \exp(-iK_n z) \int \phi_{n,-s}^{e}(r') \, I_o(r') \, dr'$$

(47a) $$V_{n,s}^{e} = V_{n,s}^{e}(o) = a_n + (i/aK_n) \int \phi_{n,-s}^{e}(r') \, I_o(r') \, dr'$$

From (32a)

$$I_{n,s}^{e}(z) = (Y_n/iK_n) \frac{d}{dz} V_{n,s}^{e}(z) = (iY_n/K_n) \Big[ K_n a_n \sin K_n z + a^{-1} \exp(-iK_n s) \cdot$$

(48) $$\cdot \int \phi_{n,-s}^{e}(r') \; I_o(r') \, dr' \Big]$$

(48a) $$I_{n,s}^{e} = I_{n,s}^{e}(o) = (iY_n/aK_n) \int \phi_{n,-s}^{e}(r') \; I_o(r') \, dr'$$

The integral equation for the problem is

(49) $$\sum_{-i}^{j} a_n \phi_{n,s}^{e}(r) = -\int G^{e}(s;r;r') \; I_o(r') \, dr'$$

where the integral is taken along the half wire lying in $z \leq 0$. The Green's function can be split into two parts as follows:

(50)
$$\begin{aligned} G^{e}(s;r;r') = (i/2a) \Big\{ & {\sum}' K_n^{-1} \exp si(k\sin\theta + 2\pi n/a)(x-x') \big[ \exp iK_n |z-z'| + \exp iK_n |z+z'| \big] \\ & + i \sum_{-i}^{j} K_n^{-1} \exp si(k \sin\theta + 2\pi n/a)(x-x') \big[ \sin K_n |z-z'| + \sin K_n |z+z'| \big] \Big\} \\ & + (i/2a) \sum_{-i}^{j} K_n^{-1} \exp si(k \sin\theta + 2\pi n/a)(x-x') \big[ \cos K_m |z-z'| + \cos K_n |z+z'| \big] \\ & = \Gamma^{e}(s;r;r') + (i/a) \sum_{-i}^{j} K_n^{-1} \phi_{n,s}^{e}(r) \; \phi_{n,-s}^{e}(r') \end{aligned}$$

where $\sum'$ denotes summation over all non-propagated modes.

The integral equation becomes then

$$\sum_{-i}^{j} \left[a_n + (1/aK_n) \int \phi^{e}_{n,-s}(r')\ I_o(r')\, dr'\right] \phi^{e}_{n,s}(r) = -\int \Gamma^{e}(s;r;r')\ I_o(r')\, dr'$$

or, from (47a)

(51) $$\sum_{-i}^{j} V^{e}_{n,s}\, \phi^{e}_{n,s}(r) = -\int \Gamma^{e}(s;r;r')\ I_o(r')\, dr'$$

and, making use of (46)

(52) $$\sum_{-i}^{j} V^{e}_{n,s}\left[\phi^{e}_{n,s}(r) + \int \Gamma^{e}(s;r;r')\, \mathcal{I}^{e}(r')\, dr'\right] = 0$$

Finally, since V's are arbitrary, one coefficient of each $V^{e}_{n,s}$ must vanish, thus giving us an integral equation for the $n^{th}$ mode current distribution.

(53) $$\phi^{e}_{n,s}(r) = -\int \Gamma^{e}(s;r;r')\, \mathcal{I}^{e}_{n,s}(r')\, dr'$$

We now obtain the admittance matrix elements by substituting

(46) into (48a)

(54) $$I^{e}_{n,s} = (iY_n/aK_n) \sum_{-i}^{j} V^{e}_{m,s} \int \phi^{e}_{n,-s}(r')\, \mathcal{I}^{e}_{m,s}(r')\, dr'$$

so that

(55) $$(Y^{11} + Y^{12})_{m,s;n,s} = (i\eta/ak) \int \phi^{e}_{n,-s}(r')\, \mathcal{I}^{e}_{m,s}(r')\, dr$$

Following the same procedure as in the derivation of stationary expressions for the scattering matrix, we multiply (53) by $\mathcal{I}^{(r)}_{m,s}$ and integrate

(56) $$-\int \phi^{e}_{m,s}(r)\, \mathcal{I}^{e}_{n,-s}(r)\, dr = \iint \mathcal{I}^{e}_{n,-s}(r)\, \Gamma^{e}(s;r;r')\, \mathcal{I}^{e}_{m,s}(r')\, dr'\, dr$$

and we obtain

$$(57)\quad (Y^{11} + Y^{12})_{m,s;n,s} = -(i\eta/ak)\ \frac{\int \phi_{n,-s}^{e}(r')\,\mathcal{J}_{m,+s}^{e}(r')\,dr' \int \phi_{m,s}^{e}(r)\,\mathcal{J}_{n,-s}^{e}(r)\,dr}{\iint \mathcal{J}_{n,-s}(r)\,\Gamma^{e}(s;r;r')\,\mathcal{J}_{m,s}(r')\,dr'\,dr}$$

It follows that the law of reciprocity holds in the usual circuit sense

$$(58)\qquad (Y^{11} + Y^{12})_{m,s;n,s} = (Y^{11} + Y^{12})_{n,-s;\,m,-s}$$

Furthermore, by considering the unperturbed field of the form $\phi_{n,s}^{e} \pm \phi_{n,-s}^{e}$, it can be shown that the admittance matrix elements are imaginary. These unperturbed fields are exactly the fields in a rectangular waveguide considered by Schwinger[6].

## Odd Case

In the same manner we evaluate not the admittance matrix, but its inverse, the impedance matrix, for the odd field. This is natural since, for no perturbation grating vanishing or absent, we have

$$Y^{11} + Y^{12} = 0$$

and (59)

$$Z^{11} - Z^{12} = 0$$

so that in the perturbation theory used here the inverse matrices would be singular in this case.

We have then

$$(60)\qquad \phi^{o}(r) = \sum_{-i}^{j} a_n\, \phi_{n,s}^{o}(r)$$

$$(61)\qquad I_o(r) = \sum_{-i}^{J} I_{n,s}^{o}\, \mathcal{J}_{n,s}^{o}(r)$$

$$(62)\quad V_{n,s}^{o}(z) = a_n \sin K_n z + (1/aK_n) \exp(-iK_n z) \int \phi_{n,-s}^{o}(r')\ I_o(r')\,dr'$$

$$(62a)\qquad V_{n,s}^{o} = V_{n,s}^{o}(0) = -(1/aK_n) \int \phi_{n,-s}^{o}(r')\ I_o(r')\,dr'$$

(63)

$$I^o_{n,s} = I^o_{n,s}(0) = (Y_n/iK_n)\frac{\partial}{\partial z}V^o_{n,s}(z)\Big|_{z=0} = -iY_n\left[a_n-(i/aK_n)\int\phi^o_{n,-s}(r')\,I_o(r')dr'\right]$$

Splitting the Green's function in the same way as in the even case

$$G^o(s;r;r')=(i/2a)\Big\{\sum{}'K_n^{-1}\exp\left[si(k\sin\theta+2\pi n/a)(x-x')\right](\exp iK_n|z-z'|-\exp iK_n|z+z'|)$$

$$+\,i\sum_{-i}^{j}K_n^{-1}\exp\left[si(k\sin\theta+2\pi n/a)(x-x')\right](\sin K_n|z-z'|-\sin K_n|z+z'|)\Big\}$$

(64)

$$(i/2a)\sum_{-i}^{j}K_n^{-1}\exp\left[si(k\sin\theta+2\pi n/a)(x-x')\right]\left[\cos K_n(z-z')-\cos K_n(z+z')\right]$$

$$=\Gamma^o(s;r;r')+(i/a)\sum_{-i}^{j}K_n^{-1}\phi^o_{n,s}(r)\,\phi^o_{n,-s}(r)$$

The integral equation is then

(65) $$\sum_{-i}^{j}\left[a_n-(i/aK_n)\int\phi^o_{n,-s}(r')\,I_o(r')\,dr'\right]\phi^o_{n,s}(r)=-\int\Gamma^o(s;r;r')\,I_o(r')\,dr'$$

From the above and (61), equating the coefficients of $I^o_{n,s}$

(66) $$\phi^o_{n,s}(r) = -i\,Y_n\int\Gamma^o(s;r;r')\,\mathcal{J}^o_{n,s}(r')\,dr'$$

From (62a)

(67) $$V^o_{n,s} = -(1/aK_n)\sum_{-i}^{j}I_{m,s}\int\phi^o_{n,-s}(r')\,\mathcal{J}^o_{m,s}(r')\,dr'$$

so that

(68) $$(Z^{11}-Z^{12})_{m,s;n,s} = -(1/aK_n)\int\phi^o_{n,-s}(r')\,\mathcal{J}^o_{m,s}(r')\,dr'$$

But, from (66)

(69) $$\int\phi^o_{m,s}(r)\,\mathcal{J}_{n,-s}(r)\,dr = -i\,Y_m\iint\mathcal{J}^o_{n,-s}(r)\,\Gamma^o(s;r;r')\,\mathcal{J}^o_{m,s}(r')\,dr'dr$$

and finally

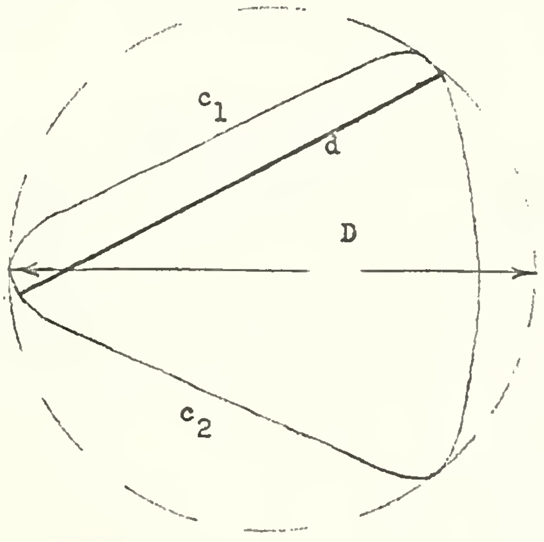
(70) $$(Z^{11}-Z^{12})_{m,s;n,s} = (i/aK_nY_m)\frac{\int\phi^o_{n,-s}(r')\,\mathcal{J}^o_{m,s}(r')dr'\int\phi^o_{m,s}(r)\,\mathcal{J}^o_{n,-s}(r)dr}{\iint\mathcal{J}^o_{n,-s}(r)\,\Gamma^o(s;r;r')\,\mathcal{J}_{m,s}(r')\,dr'\,dr}$$

The derivation of similar equations for the impedance and admittance matrices for a T M wave follows identically, and for that reason will not be presented here.

## 8. Application to the Thin Wire Grating

The impedance matrix elements will now be evaluated approximately for the case of a grating with sufficiently thin wires. It turns out that if all but the most significant term are dropped, it is possible to construct a very simple equivalent circuit. The purpose of this section is to obtain this circuit rather than to evaluate the matrix elements as accurately as possible. Furthermore, only the transverse electric field will be considered, since the diffraction effects in the transverse magnetic case are negligible.

In the case of thin wires we can use the so-called static approximation. Let us first find an asymptotic form of the Green's function $G^e$ for $r \quad r'$. $G^e$ is the radiation field of a set of line currents at $(x' + na, z')$ and their images in the $z=0$ plane. If we let $|r-r'|$ be small compared to a, the contributions of all line currents, except that at $(x', z')$ and its image, can be neglected. Furthermore, if $k|r-r'| << 1$, the function which gives the radiation field of a single current filament can be approximated by $\frac{2}{\pi} \log k|r-r'|$. Since $\Gamma^e$ differs from $G^e$ by a bounded function, this difference can be neglected compared to the logarithmic term.



Wire Cross-Section

Fig. 3

Thus we see that the integral $\int_{c_1} dr \int_{c_1} dr' \, \mathcal{J}_a(r) \, \Gamma^{e}(r,r') \, \mathcal{J}_b(r')$ can be replaced by $\frac{2}{\pi} \int_{c_1} dr \int_{c_1+c_2} dr' \, \mathcal{J}_a(r) \log|r-r'| \, \mathcal{J}_b(r')$ , $\mathcal{J}(r')$ defined on $c_2$ by reflection (see Fig. 3). Due to symmetry, we can perform the reflection with respect to r and the integral becomes $\frac{1}{\pi}\oint\oint \mathcal{J}_a(r) \log|r-r'| \, \mathcal{J}_b(r')$ . If we then let $\mathcal{J}_a(r) = \mathcal{J}_b(r) = q(r)$ the static charge distribution on a single wire in infinite space, the integral is proportional to the energy of the charge distribution. Using the theorem[14] that the introduction of an uncharged conductor into an electro-static field of fixed charges diminishes the total energy of the field, we conclude that the energy is bounded from above by that of a strip contained by the wire and from below by that of a cylinder completely enclosing the wire. These bounds are easily evaluated; in the limit of thin wire only the logarithmic term is retained, and the bounds turn out to be $Q^2 \frac{2}{\pi} \log kd$ and $Q^2 \frac{2}{\pi} \log kD$, $Q = \int_{c_1} q(r)dr$. Assuming that $\frac{D}{d}$ is bounded, the two limits differ by at most a constant, and since we have been neglecting constants compared to the logarithm, take the lower bound as the approximate value of the integral.

It remains to evaluate integrals of the type $\int_{c_1} \phi(r) \, q(r)dr$. Since the wire is thin $\phi(r)$ on $c_1$ differs from its value at $r_0$, the "center" of the wire, by a small quantity, and we can take $(r_0)$ as a mean value. The value of the integral is then $Q\,\phi(r_0)$.

We have then

$$(71) \qquad (Y_{11} + Y_{12})_{m,s;n,s} = i \, \frac{\pi\eta}{2ak} \, \frac{\phi_{m,s}(r_0) \, \phi_{n,-s}(r_0)}{\log kD} = i \, \frac{\pi\eta}{2ak} \, \frac{1}{\log kD}$$

taking $r_0 = 0$ .

---

[14] Stratton, Electromagnetic Waves, New York, 1941, p. 117

For thin wires the transverse electric fields in the propagated modes are very nearly continuous across the z=0 plane so that $(Z_{11} - Z_{12}) = 0$, and

$$I^1 + I^2 = 2(Y^{11} + Y^{12})\ V \tag{72}$$

where the superscript of V was dropped since $V^1 = V^2$. This means that the grating can be represented by a shunt network (see Fig. 4) in which the shunt current is given by $(i\pi\eta/2ak \log kD)(\sum V_{m,s})$. Such a network is shown on Fig. 3. The transformation ratios are equal, but arbitrary, and we can set

$$N = 1 \tag{73a}$$

The impedance is then

$$Z = -i\,\frac{2ak \log kD}{\pi\eta} \tag{73b}$$

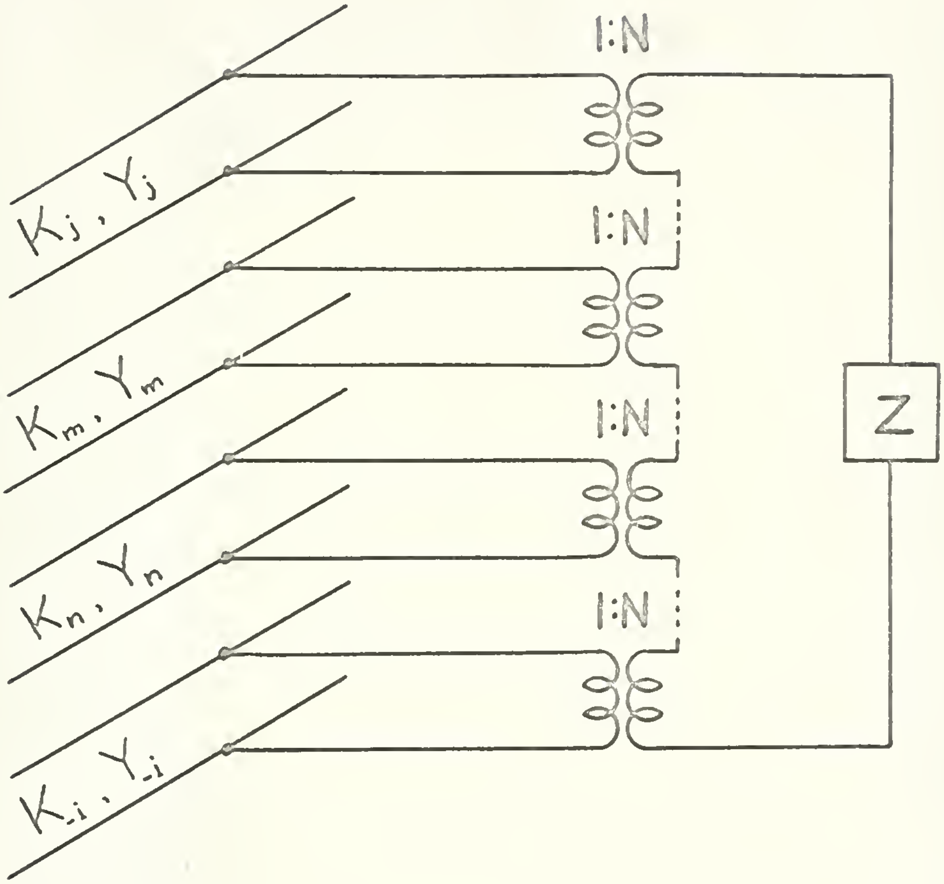


Fig. 4

Equivalent Network of Thin Wire Grating

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